THE CHINESE UNIVERSITY OF HONG KONG Department of Mathematics

MATH2060B Mathematical Analysis II Homework 9 Suggested Solutions

1. (Exercise 8.2.15 of [BS11]) Let $g_n(x) := nx(1-x)^n$ for $x \in [0,1], n \in \mathbb{N}$. Discuss the convergence of (g_n) and $(\int_0^1 g_n dx)$.

Solution. Note first that when x = 0, $g_n(0) = 0$ for all $n \in \mathbb{N}$. Now let $x \in (0, 1]$. Then we employ the ratio test (Theorem 3.2.11 of [BS11]). We have

$$\frac{g_{n+1}(x)}{g_n(x)} = \frac{(n+1)x(1-x)^{n+1}}{nx(1-x)^n} = \frac{(n+1)(1-x)}{n}$$
$$= 1 - x + \frac{1}{n}(1-x) \to 1 - x < 1 \text{ as } n \to +\infty$$

and hence by the Ratio test, we conclude that $g_n(x) \to 0$ as $n \to +\infty$ for all $x \in (0,1]$.

For the convergence of $(\int_0^1 g_n(x)dx)$, we show that g_n is uniformly bounded on [0,1] for all $n \in \mathbb{N}$ and use the Bounded Convergence Theorem (Theorem 8.2.5 of [BS11]). In the first derivative test, we have

$$0 = g'_n(x) = n(1-x)^n - n^2x(1-x)^{n-1} \Rightarrow x = \frac{1}{n+1}$$

and using the second derivative, we have

$$g_n''(x) = -2n^2(1-x)^{n-1} + n^2(n-1)x(1-x)^{n-2}$$

$$g_n''(1/(n+1)) = -2n^2\left(\frac{n}{n+1}\right)^{n-1} + n^2(n-1)\left(\frac{1}{n+1}\right)\left(\frac{n}{n+1}\right)^{n-2}$$

$$= -(1+n)n\left(\frac{n}{n+1}\right)^{n-1} < 0$$

and hence we see that $g_n(x)$ achieves maximum on [0,1] at $x=\frac{1}{n+1}$ with value

$$g_n\left(\frac{1}{n+1}\right) = n\left(\frac{1}{n+1}\right)\left(1 - \frac{1}{n+1}\right)^n = \left(\frac{n}{n+1}\right)^n \le 1.$$

Moreover, since $g_n(x) \geq 0$ on [0,1], we see that $|g_n(x)| \leq 1$ for all $n \in \mathbb{N}$ and $x \in [0,1]$ and so by the Bounded Convergence Theorem, we conclude that

$$\lim_{n \to +\infty} \int_0^1 g_n(x) dx = \int_0^1 \lim_{n \to +\infty} g_n(x) dx = \int_0^1 0 dx = 0.$$

2. (Exercise 8.2.17 of [BS11]) Let $f_n(x) := 1$ for $x \in (0, 1/n)$ and $f_n(x) := 0$ elsewhere in [0, 1]. Show that (f_n) is a decreasing sequence of discontinuous functions that converges to a continuous limit function, but the convergence is not uniform on [0, 1].

Solution. It is clear that each f_n is discontinuous at x = 1/n. Observe that

$$f_n(x) - f_{n+1}(x) = \begin{cases} 0, & 0 \le 0 < \frac{1}{n+1} \\ 1, & \frac{1}{n+1} \le x < \frac{1}{n} \Rightarrow f_n(x) - f_{n+1}(x) \ge 0, x \in [0, 1] \\ 0, & \frac{1}{n} \le x \le 1 \end{cases}$$

and hence f_n is a decreasing sequence of functions. We show that f_n converges to f(x) := 0 for $x \in [0,1]$: Note that $f_n(0) = 0$ for all $n \in \mathbb{N}$. Now let $x_0 \in (0,1]$ be given. Then we can find an $N \in \mathbb{N}$ such that $\frac{1}{N} < x_0$ and hence for all $n \geq N$, $|f_n(x_0)| = 0$. So f_n converges pointwise to the zero function, which is continuous on [0,1], since it is constant.

On the other hand, we show that we can find a sequence $(x_n) \subset [0,1]$ such that $|f_n(x_n) - f(x_n)| \ge 1$ to show that this convergence is not uniform. Let $x_n = \frac{1}{2n}$, then we see that since $0 < \frac{1}{2n} < \frac{1}{n}$, $|f_n(\frac{1}{2n})| = 1$ for all $n \in \mathbb{N}$, as required.

3. (Exercise 8.3.8 of [BS11]) Let $f : \mathbb{R} \to \mathbb{R}$ be such that f'(x) = f(x) for all $x \in \mathbb{R}$. Show that there exists $K \in \mathbb{R}$ such that $f(x) = Ke^x$ for all $x \in \mathbb{R}$.

Solution. We split into two cases. We first consider the case when $f(0) \neq 0$. Then g(x) := f(x)/f(0) is well-defined for $x \in \mathbb{R}$ and we verify the following:

$$g'(x) = \frac{f'(x)}{f(0)} = \frac{f(x)}{f(0)} = g(x),$$

$$g(0) = \frac{f(0)}{f(0)} = 1.$$

Then g(x) satisfies the properties of the function in Theorem 8.3.1 of [BS11] and by the uniqueness theorem (Theorem 8.3.4 of [BS11]), we have that $g(x) = e^x$. Hence, we have that $f(x) = f(0)e^x$ and so we take K = f(0).

The second case is if f(0) = 0. Then we will show f(x) = 0 using the argument in the proof of Theorem 8.3.4 of [BS11]. By induction, $f^{(n)}(x)$ exists on \mathbb{R} for all $n \in \mathbb{N}$ and equals f(x): the base case is given by the assumption in the question; if $f^{(n)}(x)$ exists on \mathbb{R} and equals f(x), then $f^{(n+1)}(x) = (f^{(n)}(x))' = (f(x))' = f'(x) = f(x)$ and hence $f^{(n+1)}(x)$ exists on \mathbb{R} and equals f(x). Let $x \in \mathbb{R}$ be given. Then f(t) is bounded on [0, x], that is, there is an M > 0 such that $|f(t)| \leq M$ for all $t \in [0, x]$. Then by Taylor expansion, there is a $c_n \in [0, x]$ such that

$$|f(x)| = \left| f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \dots + \frac{f^{(n-1)}(0)}{(n-1)!} x^{n-1} + \frac{f^{(n)}(c_n)}{n!} x^n \right|$$
$$= \left| \frac{f^{(n)}(c_n)}{n!} x^n \right| \le \frac{M}{n!} |x|^n$$

since this is true for all $n \in \mathbb{N}$ and $\lim_{n \to +\infty} \frac{M}{n!} |x|^n = 0$, we have that f(x) = 0. Hence, we can take K = 0 in this case.

4. (Exercise 8.3.9 of [BS11]) Let $a_k > 0$ for k = 1, ..., n and let $A := (a_1 + \cdots + a_n)/n$ be the arithmetic mean of these numbers. For each k, put $x_k := a_k/A - 1$ in the inequality $1 + x \le e^x$. Multiply the resulting terms to prove the Arithmetic-Geometric Mean Inequality

$$(a_1 \cdots a_n)^{1/n} \le \frac{1}{2}(a_1 + \cdots + a_n).$$
 (1)

Moreover, show that the equality holds in 1 if and only if $a_1 = a_2 = \cdots = a_n$.

Solution. Defining x_k as in the question, and following the suggested argument, we see that

$$1 + x_k \le e^{x_k}$$
 for each $k = 1, \dots, n$.

Multiplying all of these inequalities together, we obtain

$$\prod_{k=1}^{n} (1+x_k) \le e^{x_1} e^{x_2} \cdots e^{x_k} = \exp\left(\sum_{k=1}^{n} x_k\right)$$

$$= \exp\left(\frac{a_1 + a_2 + \dots + a_n}{A} - n\right) = \exp\left(\frac{nA}{A} - n\right) = e^0 = 1.$$

The left-hand side simplifies to $\prod_{k=1}^{n} (1 + x_k) = \prod_{k=1}^{n} \frac{a_k}{A}$ and hence

$$\prod_{k=1}^{n} \frac{a_k}{A} \le 1 \Rightarrow \prod_{k=1}^{n} a_k \le A^n \Rightarrow \left(\prod_{k=1}^{n} a_k\right)^{1/n} \le A$$

which is the required Arithmetic-Geometric Mean inequality. Note that equality holds in the product inequality above if and only if both sides of the inequality is 1, that is, $x_k = 0$ for each k. Then this gives $x_k = 1 \Leftrightarrow a_k = A$ for each $k = 1, \ldots, n$, that is, if and only if $a_1 = a_2 = \cdots = a_n = A$.

5. (Exercise 9.1.6 of [BS11]) Find an explicit expression for the *n*th partial sum of $\sum_{n=2}^{\infty} \ln(1-1/n^2)$ to show that this series converges to $-\ln 2$. Is this convergence absolute?

Solution. Let $n \geq 2$ be fixed. We consider the *n*-th partial sum:

$$s_n = \sum_{k=2}^n \ln\left(1 - \frac{1}{k^2}\right) = \sum_{k=2}^n \ln\left(\frac{k^2 - 1}{k^2}\right) = \sum_{k=2}^n \ln(k - 1) + \ln(k + 1) - 2\ln(k)$$

using logarithmic identities. We prove using induction that this sum is equal to $-\ln(2) - \ln(n) + \ln(n+1)$ for each n. Note that the base case of n=2 is trivial. Suppose $s_n = -\ln(2) - \ln(n) + \ln(n+1)$ for some n. Then by above,

$$s_{n+1} = s_n + \ln(n) + \ln(n+2) - 2\ln(n+1)$$

= $-\ln(2) - \ln(n) + \ln(n+1) + \ln(n) + \ln(n+2) - 2\ln(n+1)$
= $-\ln(2) - \ln(n+1) + \ln(n+2)$

which is the desired formula for s_{n+1} and hence the explicit formula for the n-th partial sum is

$$s_n = -\ln(2) - \ln(n) + \ln(n+1).$$

We prove that the sum converges to $-\ln(2)$. By the continuity of $\ln(x)$, we have

$$\lim_{n \to \infty} s_n = -\ln(2) + \lim_{n \to \infty} \ln\left(\frac{n+1}{n}\right) = -\ln(2) + \ln\left(\lim_{n \to \infty} \frac{n+1}{n}\right)$$
$$= -\ln(2) + \ln(1) = -\ln(2).$$

Note that for $n \geq 2$, $1 - \frac{1}{n^2} \leq 1$ and hence each summand $\ln\left(1 - \frac{1}{n^2}\right) \leq 0$. Therefore, to show that the series is absolutely convergent, we need to show that $\sum_{n=2}^{\infty} \left|\ln\left(1 - \frac{1}{n^2}\right)\right| = \sum_{n=2}^{\infty} -\ln\left(1 - \frac{1}{n^2}\right) \text{ converges. A similar induction argument shows that each partial sum in this new series is simply}$

$$-s_n = \ln(2) + \ln(n) - \ln(n+1) = \ln(2) + \ln\left(\frac{n}{n+1}\right) \to \ln(2) \text{ as } n \to \infty.$$

Therefore, we conclude that the series is absolutely convergent.

6. (Exercise 9.1.12 of [BS11]) Let a > 0. Show that the series $\sum (1+a^n)^{-1}$ is divergent if $0 < a \le 1$ and is convergent if a > 1.

Solution. We make use of the fact that if the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. Taking contrapositive, if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges (in the textbook, this is called the *n*-th Term Test, Theorem 3.7.3 of [BS11]). When 0 < a < 1, $a^n \to 0$ as $n \to \infty$ and hence $\frac{1}{1+a^n} \to 1 \neq 0$ as $n \to \infty$ and so by above, the series diverges. Similarly, when a = 1, $\frac{1}{1+a^n} = \frac{1}{2}$ and so the series diverges. When a > 1, we have

$$\frac{1}{1+a^n} \le \frac{1}{a^n} = \left(\frac{1}{a}\right)^n, \quad \text{and} \quad \sum_{n=1}^{\infty} \left(\frac{1}{a}\right)^n = \frac{1}{1-1/a}$$

the geometric series with ratio 0 < 1/a < 1. Hence by the comparison test, $\sum_{n=1}^{\infty} \frac{1}{1-a^n}$ converges in this case.

References

[BS11] Robert G. Bartle and Donald R. Sherbert. *Introduction to Real Analysis, Fourth Edition*. Fourth. University of Illinois, Urbana-Champaign: John Wiley & Sons, Inc., 2011. ISBN: 978-0-471-43331-6.